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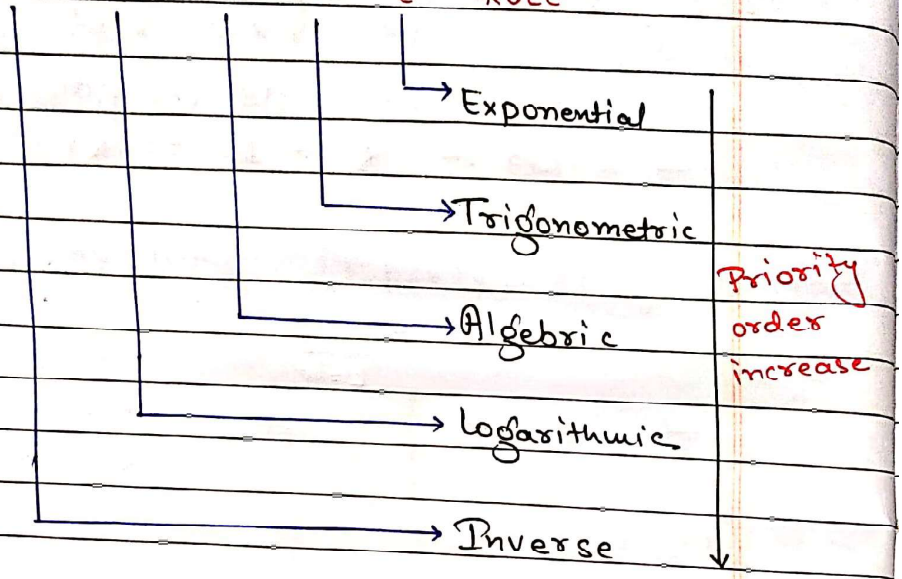
For Integrating by Parts :

I L A T E RULE



* $\int \cos nx \cdot dx = \frac{\sin nx}{n}$

* $\int \sin nx \cdot dx = \frac{-\cos nx}{n}$

* $\sin 0 = 0$ * $\sin 2n\pi = 0$

* $\cos 0 = 1$ * $\cos 2n\pi = 1$

* $\sin n\pi = 0$ * $\sin(-n\pi) = 0$

* $\cos n\pi = (-1)^n$ * $\cos(-n\pi) = (-1)^n$

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FOURIER'S SERIES.

A series consisting of only the terms of sine and cosine.

$f(x)$ = given + limits } Fourier's series.

Sequence: A list of numbers separated by commas(,).

for e.g; 2, 4, 6, 8, 10, 12 ---

Series: A list of numbers separated by (+) or (−).

for e.g. 2 + 4 − 6 + 8 − 10 + 12 ---

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

(find) (Given)

Fourier's series.

$$f(x) = a_0 \cos 0x + a_1 \cos 1x + a_2 \cos 2x + \cdots$$

$$+ b_0 \sin 0x + b_1 \sin 1x + b_2 \sin 2x + \cdots$$

or,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Euler's theorem...

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\beta} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\beta} f(x) \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\beta} f(x) \sin nx \cdot dx$$

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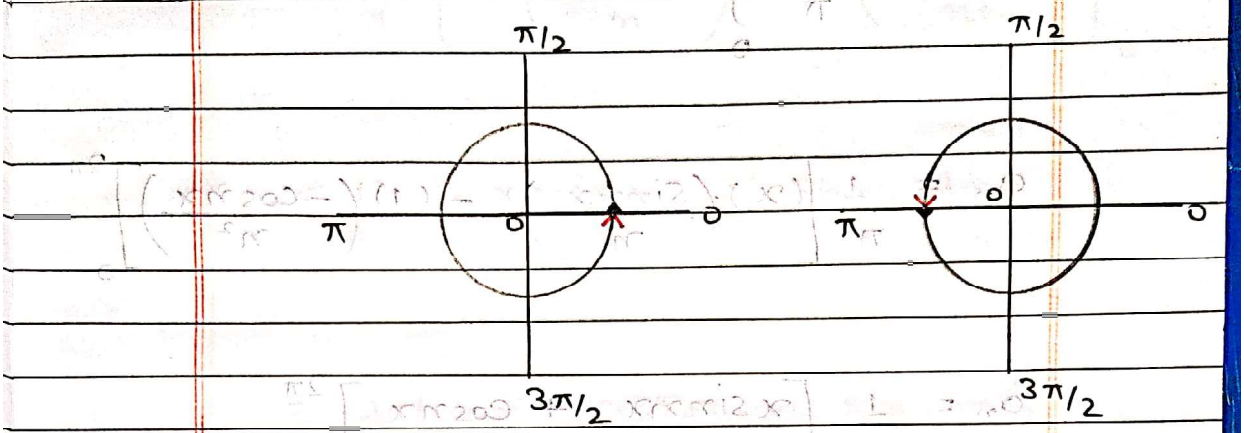
Full-range Series

limit → $(0, 2\pi)$ & $(-\pi, \pi)$



Q. Express $f(x) = x$, $(0, 2\pi)$ as Fourier's series:

Sol. Given: $f(x) = x$, $(0, 2\pi)$

As we know,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Now,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x \cdot dx$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left[\frac{(2\pi)^2}{2} - 0\right]$$

$$\Rightarrow a_0 = \frac{1}{\pi} \frac{4\pi^2}{2} = 2\pi$$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \cdot \cos nx \cdot dx$$

$$a_n = \frac{1}{\pi}\left[(x)\cdot\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\left\{\frac{2\pi\cdot\sin 2\pi}{n} + \frac{\cos n2\pi}{n^2}\right\} - \left\{\frac{0\cdot\sin n(0)}{n} + \frac{\cos n(0)}{n^2}\right\}\right]$$

$$a_n = \frac{1}{\pi}\left[0 + \frac{1}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{1}{\pi} \times 0$$

$$a_n = 0$$

Now,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \cdot \sin nx \cdot dx$$

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$$b_n = \frac{1}{\pi}\left[x\cdot\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-2\pi\cdot\cos 2n\pi}{n} + \frac{\sin 2n\pi}{n^2}\right\} - \left\{\frac{-0\cdot\cos n(0)}{n} + \frac{\sin n(0)}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left(\frac{-2\pi}{n}\right)$$

$$b_n = \frac{-2}{n}$$

Now, fourier series is given by

$$\Rightarrow f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty}(0)\cdot\cos nx + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\cdot\sin nx$$

$$\Rightarrow f(x) = \pi + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\cdot\sin nx$$

$$\Rightarrow f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

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Q. Express $f(x) = x^2$, $(-\pi, \pi)$ as fourier's series.

Sol.

Given: $f(x) = x^2$ $(-\pi, \pi)$

By Euler's theorem

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Now,

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot dx$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$\Rightarrow a_0 = \frac{2\pi^2}{3}$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cdot\cos nx\cdot dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2\cdot\sin nx}{n} + \frac{2x\cdot\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi^2\sin n\pi}{n} + \frac{2(\pi)\cdot\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right\} - \left\{\frac{(-\pi)^2\cdot\sin n(-\pi)}{n} + \frac{2(-\pi)\cdot\cos n(-\pi)}{n^2} - \frac{2\sin n(-\pi)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left\{0 + \frac{2\pi\cdot\cos n\pi}{n^2} - 0\right\} - \left\{0 - \frac{2\pi\cos n\pi}{n^2} - 0\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi(-1)^n}{n^2} + \frac{2\pi(-1)^n}{n^2}\right]$$

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$$a_n = \frac{1}{\pi} \cdot \frac{4\pi(-1)^n}{n^2}$$

$$\Rightarrow a_n = \frac{4(-1)^n}{n^2}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi}\left[(x^2)\cdot\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cdot\cos nx}{n} + \frac{2x\cdot\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\pi^2\cdot\cos n\pi}{n} + \frac{2\pi\cdot\sin n\pi}{n^2} + \frac{2\cos n\pi}{n^3}\right\}\right.$$

$$\left. - \left\{\frac{-(-\pi)^2\cdot\cos n(-\pi)}{n} + \frac{2(-\pi)\cdot\sin n(-\pi)}{n^2} + \frac{2\cos n(-\pi)}{n^3}\right\}\right]$$

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$$b_n\pi = \frac{1}{\pi}\left[\left\{\frac{-\pi^2\cdot(-1)^n}{n} + 0 + \frac{2(-1)^n}{n^3}\right\} - \left\{\frac{-\pi^2\cdot(-1)^n}{n} + 0 + \frac{2(-1)^n}{n^3}\right\}\right]$$

$$b_n = 0$$

Now, Fourier's series is given by

$$f(x) = \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cdot\cos nx + \sum_{n=1}^{\infty}(0)\cdot\sin nx.$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot\cos nx.$$

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$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

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Q. Express $f(x) = e^{-ax}$, $(-\pi, \pi)$ as fourier's series.

Sol. Given $f(x) = e^{-ax}$ $(-\pi, \pi)$

As we know, by Euler's theorem

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Now,

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_0 = \frac{-1}{a\pi}\left[e^{-a\pi} - e^{a\pi}\right]$$

$$\Rightarrow a_0 = \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right]$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cdot\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

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$$\int e^{ax} \sin bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\left\{\frac{e^{-a\pi}}{a^2+n^2}(n\sin n\pi - a\cos n\pi)\right\} - \left\{\frac{e^{+a\pi}}{a^2+n^2}(n\sin n(-\pi) - a\cos n(-\pi))\right\}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\left\{\frac{e^{-a\pi}}{a^2+n^2}(-a(-1)^n)\right\} - \left\{\frac{e^{a\pi}}{a^2+n^2}(-a(-1)^n)\right\}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi(a^2+n^2)}\left[-e^{-a\pi}a(-1)^n + e^{a\pi}a(-1)^n\right]$$

$$\Rightarrow a_n = \frac{1}{(a^2+n^2)\pi}\left[ae^{a\pi}(-1)^n - ae^{-a\pi}(-1)^n\right]$$

$$\Rightarrow a_n = \frac{a(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right]$$

and,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \cdot \sin nx \cdot dx$$

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$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(a^2+n^2)}\left(-a\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$\Rightarrow = \frac{1}{\pi}\left[\left\{\frac{e^{-a\pi}}{(a^2+n^2)}\left(-a\sin n\pi - n\cos n\pi\right)\right\}\right.$$

$$\left.-\left\{\frac{e^{+a\pi}}{a^2+n^2}\left(-a\sin(-n\pi) - n\cos n(-\pi)\right)\right\}\right]$$

$$\Rightarrow = \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}\left\{-n(-1)^n\right\} - e^{a\pi}\left\{-n(-1)^n\right\}\right]$$

$$\Rightarrow = \frac{1}{\pi(a^2+n^2)}\left[-e^{-a\pi}n(-1)^n - e^{a\pi}n(-1)^n\right]$$

$$\Rightarrow = \frac{-n(-1)^n}{\pi(a^2+n^2)}\left[e^{-a\pi} + e^{a\pi}\right]$$

$$\Rightarrow b_n = \frac{-n(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi} + e^{-a\pi}\right]$$

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Now, Fourier series is Given by.

$$f(x) = \frac{(e^{a\pi} - e^{-a\pi})}{2a\pi} + \left\{ \sum_{n=1}^{\infty} \frac{a(-1)^n (e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} \cos nx \right\} + \left\{ \sum_{n=1}^{\infty} \frac{(-n)(-1)^n}{\pi(a^2+n^2)} (e^{a\pi} + e^{-a\pi}) \sin nx \right\}$$

Q. Express $f(x) = x \sin x$, $(0, 2\pi)$ to Fourier's series.

Sol.. Here $f(x) = x \sin x$, $(0, 2\pi)$

By Euler's theorem,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Now,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x \sin x \, dx$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left[x \cdot (-\cos x) - (1)(-\sin x) \right]_0^{2\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left[-x \cos x + \sin x \right]_0^{2\pi}$$

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$2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

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$$\frac{1}{\pi}\left[\sin x - x\cos x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\{\sin 2\pi - 2\pi\cos 2\pi\} - \{\sin 0 - 0\cdot\cos 0\}\right]$$

$$= \frac{1}{\pi}\left[-2\pi(1)\right]$$

$$= -2$$

Now,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cdot \cos nx \cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cdot\cos nx \cdot \sin x \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\cos nx \cdot \sin x \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(nx+x) - \sin(nx-x)\right] dx$$

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$$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]\cdot dx$$

$$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} x\cdot\sin(n+1)x\cdot dx - \frac{1}{2\pi}\int_0^{2\pi} x\cdot\sin(n-1)\cdot dx$$

By Integration by Parts.

$$\Rightarrow a_n = \frac{1}{2\pi}\left[x\cdot\left\{\frac{-\cos(n+1)x}{(n+1)}\right\} - (1)\left\{\frac{-\sin(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi} -$$

$$\frac{1}{2\pi}\left[x\cdot\left\{\frac{-\cos(n-1)x}{(n-1)}\right\} - (1)\left\{\frac{-\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[\frac{-x\cos(n+1)x}{(n+1)} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi} -$$

$$\frac{1}{2\pi}\left[\frac{-x\cos(n-1)x}{(n-1)} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[\frac{(-2\pi)(1)}{(n+1)} + 0\right] - \frac{1}{2\pi}\left[\frac{(2\pi)(1)}{(n-1)} + 0\right]$$

$$\Rightarrow a_n = \frac{-1}{(n+1)} - \frac{1}{(n-1)}$$

$$\Rightarrow a_n = \frac{+2}{(n^2-1)}$$

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$2\sin x \cdot \sin y = \cos(x-y) - \cos(x+y)$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cdot \frac{2}{2}\sin nx \cdot \sin x \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin nx \cdot \sin x \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(nx-x) - \cos(nx+x)\right] dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$\therefore b_n = \frac{1}{2\pi}\int_0^{2\pi} x \cdot \cos(n-1)x \cdot dx -$$

$$\frac{1}{2\pi}\int_0^{2\pi} x \cdot \cos(n+1) \cdot x \cdot dx.$$

After solving

$$b_n = 0$$

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∴ The fourier's series is Given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cdot dx + \sum_{n=1}^{\infty} b_n \sin nx \cdot dx$$

$$\Rightarrow x \sin x = \frac{a_0}{2} + a_1 \cos 1x + a_2 \cos 2x + a_3 \cos 3x + \text{---} + b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x \text{ ---}$$

Now,

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos x \cdot dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot \cos x \cdot dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \cdot 2 \sin x \cdot \cos x \cdot dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \cdot \sin 2x \cdot dx$$

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$$= \frac{1}{2\pi} \int_0^{2\pi} x \cdot \sin 2x \cdot dx$$

$$= \frac{1}{2\pi} \left[\frac{x(-\cos 2x)}{2} - \frac{(1)(-\sin 2x)}{(2)^2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{-2\pi \cos(2\times 2\pi)}{2} + \frac{\sin 2(2\pi)}{4} \right]$$

$$= \frac{1}{2\pi} \left[\frac{-2\pi}{2} \right]$$

$$\Rightarrow a_1 = \frac{-1}{2}$$

and $$b_1 = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin x \cdot \sin x \cdot dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin^2 x \cdot dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \left(1 - \cos 2x\right) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \cdot dx - \frac{1}{2\pi} \int_0^{2\pi} x \cos 2x \cdot dx$$

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$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} - \frac{1}{\pi}\left[x\cdot\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-\cos 2x}{(2)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{4\pi x}{2}\right] - \frac{1}{\pi}\left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \pi - \frac{1}{\pi}\left[\frac{2\pi\cdot\sin 2(2\pi)}{2} + \frac{\cos 2(2\pi)}{4}\right]$$

$$= \pi - \frac{1}{2\pi}\left[\left\{\frac{1}{4} - \frac{1}{4}\right\}\right]$$

$$\Rightarrow b_1 = \pi$$

Now, fouriex's seriex is;

$$x\sin x = \frac{-2}{2} + \left(\frac{-1}{2}\right)\cos x + \frac{2}{2^2-1}\cos 2x +$$

$$\frac{2}{3^2-1}\cos 3x + - - - -$$

$$+ \pi\sin x + - - - -$$

hence, $x\sin x = -1 + \pi\sin x - \frac{1}{2}\cdot\cos x + \frac{2}{2^2-1}\cdot\cos 2x +$

$$\frac{2}{3^2-1}\cos 3x + - - -$$

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Q. Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$,

↑ previously solved.

$-\pi < x < \pi$.

Hence show that

(i). $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

(ii). $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(iii). $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Sol. Now As we know that

(iii). $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$

$= \frac{\pi^2}{3} + 4\left[-\frac{\cos 1x}{1} + \frac{\cos 2x}{(2)^2} - \frac{\cos 3x}{(3)^2} + \frac{\cos 4x}{(4)^2} - \frac{\cos 5x}{(5)^2} + \cdots\right]$ → (1)

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Put $x=0$ in eq (1)

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1} + \frac{1}{(2)^2} - \frac{1}{(3)^2} + \frac{1}{(4)^2} - \cdots\right]$$

$$\frac{\pi^2}{3} = 4\left[\frac{1}{1} - \frac{1}{(2)^2} + \frac{1}{(3)^2} - \frac{1}{(4)^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{(2)^2} + \frac{1}{(3)^2} - \frac{1}{(4)^2} - \cdots \quad \text{— (11)}$$

(i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \cdots = \frac{\pi^2}{6}$$

Put $x = \pi$ in eq (1)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{(1)^2} + \frac{1}{(2)^2} + \frac{1}{(3)^2} + \frac{1}{(4)^2} - \cdots\right]$$

$$\frac{3\pi^2 - \pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{2\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{—}\right]$$

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$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{(iii)}$$

(ii) $$\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \infty = \frac{\pi^2}{8}$$

By adding eq (ii) and (iii), we get

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \cdots \infty = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$\Rightarrow 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{7^2} + \cdots \infty\right) = \frac{3\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{7^2} + \cdots \infty = \frac{\pi^2}{8}$$

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FUNCTIONS WITH POINTS OF DISCONTINUITY:

$f(x)$ = multivalued.

Q. $f(x) = \begin{cases} x & 0 < x < \pi \\ x^2 & \pi < x < 2\pi \end{cases}$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx$$

$$= \frac{1}{\pi} \int_0^{\pi} f(x) \cdot dx + \frac{1}{\pi} \int_{\pi}^{2\pi} f(x) \cdot dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cdot dx + \frac{1}{\pi} \int_{\pi}^{2\pi} x^2 \cdot dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi} \left[\frac{x^3}{3}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2}\right] + \frac{1}{\pi} \left[\frac{8\pi^3}{3} - \frac{\pi^3}{3}\right]$$

$$\Rightarrow a_0 = \frac{\pi}{2} + \frac{7\pi^2}{3}$$

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$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos nx \cdot dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} f(x) \cdot \cos nx \cdot dx + \int_{\pi}^{2\pi} f(x) \cdot \cos nx \cdot dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \cdot \cos nx \cdot dx + \int_{\pi}^{2\pi} x^2 \cdot \cos nx \cdot dx \right]$$

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Q. $f(x) = \begin{cases} x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$

Prove that

Put $x = 0$

$f(x) = ?$

$$f(0) = \frac{f(0^-) + f(0^+)}{2}$$

Put $x = 0$

$f(x) = ?$

$$\Rightarrow f(0) = \frac{0 + \pi}{2} = \frac{\pi}{2}$$

Point of discontinuity (a)

$$f(a) = \frac{f(a^-) + f(a^+)}{2}$$

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CHANGE OF INTERVAL

$(-c, c)$ or $(0, 2c)$

where c is constant.

① Value of c

② Wherever $\frac{1}{\pi} = 1/c$

③ $nx \longrightarrow \frac{n\pi x}{c}$

for e.g. $(-4, 4)$ or $(0, 4)$

$\Downarrow$ $\Downarrow$

$(-c, c)$ $(0, 2c)$

$\Downarrow$ $\Downarrow$

$c = 4$ $c = 2$

$$a_0 = \frac{1}{c} \int_{\alpha}^{\beta} f(x) \cdot dx$$

$$a_n = \frac{1}{c} \int_{\alpha}^{\beta} f(x) \cdot \cos\frac{n\pi x}{c} \cdot dx$$

$$b_n = \frac{1}{c} \int_{\alpha}^{\beta} f(x) \cdot \sin\frac{n\pi x}{c} \cdot dx$$

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$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{c} + \sum b_n \sin\frac{n\pi x}{c}$$

EVEN OR ODD FUNCTION

Even function: $f(-x) = f(x)$

e.g. $f(x) = x^2$

$$f(-x) = (-x)^2 = x^2$$

e.g. $f(x) = x \sin x$

$$f(-x) = -x \cdot \sin(-x)$$
$$= x \sin x$$

Odd function:

e.g. $f(x) = x^3$

$$f(-x) = (-x)^3$$
$$f(-x) = -x^3$$

e.g. $f(x) = x \cos x$

$$f(-x) = -x \cos(-x)$$
$$= -x \cos x$$

Neither even nor odd:

$$f(x) = x + x^2$$
$$f(-x) = (-x) + (-x)^2$$
$$= -x + x^2.$$

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limit $(-\pi, \pi)$ or $(-c, c)$

Even	Odd	Neither even nor odd
$a_0 = ?$	$a_0 = 0$	$a_0 = ?$
$a_n = ?$	$a_n = 0$	$a_n = ?$
$b_n = 0$	$b_n = ?$	$b_n = ?$

Q. $f(x) = x^2 \quad (-\pi, \pi)$

Now, $f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$f(-x) = f(x)$

Since, it is a even function

$b_n = 0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cdot dx$$

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HALF-RANGE SERIES

SINE SERIES:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx$$

COSINE SERIES:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

$$a_0 = \frac{2}{c} \int_0^c f(x) \cdot dx$$

$$a_n = \frac{2}{c} \int_0^c f(x) \cdot \cos \frac{n\pi x}{c} \cdot dx.$$

Q. Express $f(x) = x$ as a half-range sine series in $0 < x < 2$.

Sol. Given; $f(x) = x$ $(0, 2)$

By Half Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

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$$b_n = \frac{2}{c}\int_0^2 f(x)\cdot \sin\frac{n\pi x}{c}\cdot dx$$

$$= \frac{2}{c}\int_0^2 x\cdot \sin\frac{n\pi x}{c}\cdot dx$$

$$= \frac{2}{2}\int_0^2 x\cdot \sin\frac{n\pi x}{2}\cdot dx$$

$$= \left[x\cdot\frac{\left(-\cos\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - 1\cdot\frac{\left(-\sin\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{2^2}}\right]_0^2$$

$$= \left[\frac{-x\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right]_0^2$$

$$= \left[\frac{-2\cos\left(\frac{n\pi 2}{2}\right)}{\frac{n\pi}{2}} + \frac{\sin\left(\frac{n\pi 2}{2}\right)}{\frac{n^2\pi^2}{4}}\right]$$

$$= -\frac{4(-1)^n}{n\pi}$$

$$b_n = -\frac{4(-1)^n}{n\pi}$$

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Now,

$$b_1 = \frac{-4(-1)^1}{(1)\pi}$$

$$b_2 = \frac{-4(-1)^2}{2\pi}$$

$$b_3 = \frac{-4(-1)^3}{3\pi} \quad \text{and so on.}$$

Now, the Half - sine fourier series of $f(x) = x$ is given by.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}.$$

$$= \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n\pi} \cdot \sin \frac{n\pi x}{2}$$

$$f(x) = \frac{4}{\pi} \sin \frac{\pi x}{2} - \frac{4}{2\pi} \sin \frac{2\pi x}{2} + \frac{4}{3\pi} \sin \frac{3\pi x}{2} - \cdots \infty$$

$$= 4\left(\frac{1}{\pi} \sin \frac{\pi x}{2} - \frac{1}{2\pi} \sin \frac{2\pi x}{2} + \frac{1}{3\pi} \sin \frac{3\pi x}{2} - \cdots \infty\right)$$

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Q. Express $f(x) = x$ as a half-cosine series in $0 < x < 2$.

Sol. $a_0 = 2$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$\therefore$

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} - - -\right]$$

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